Unit - II

MODULES

Vector Space:- Let 'V' be the non-empty set and $(+,\cdot)$ be two b-ary operations defined on 'V' over the field 'F' then 'V' is called vector space over the field 'F' if it satisfies.

i, $(V,+)$ is an abelian group.

ii, $(V,\cdot)$ is a semi group

iii, a $\alpha(x+y) = \alpha x + \alpha y$

b, $\alpha+\beta(x) = \alpha x + \beta x$

c, $\alpha(\beta(x)) = (\alpha\beta)(x)$

d, $1\cdot x = x \ \forall 1, \alpha, \beta \in F \ \& \ x, y \in V$

Module:- Let 'A' be the commutative ring with unity a non-empty set 'M' is called A-module if it has two binary operations

$+ : M \times M \to M$ given by $(x,y) \to x+y$

$\cdot : A \times M \to M$ given by $(a,x) \to ax$

And also it satisfies the following conditions.

i, $(M,+)$ is an abelian group.

ii, $(M,\cdot)$ is a semi group.

iii, a, $\alpha(x+y) = \alpha x + \alpha y$

b, $(\alpha+\beta)(x) = \alpha x + \beta x$

c, $(\alpha\beta)(x) = \alpha(\beta x)$

d, $1\cdot x = x \ \forall \alpha, \beta, 1 \in A \ \& \ x, y \in M$

Example:- 1, Every vector space 'V' over the field 'F' is a F-module.

2, Every abelian group 'G' is a Z-module.

Sub Module:- Let 'M' be a A-module a non-empty subset 'N' of 'M' is called sub-module if

1, $x, y \in N \Rightarrow x - y \in N$

ii, $x \in N, a \in A \Rightarrow xa \in N$.

Module-homomorphism:- Let M and M' be A-modules a mapping $f : M \to M'$ is called A-module homomorphism if it satisfies

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iii) $f(1)=1$ (or) $f(ax+by) = af(x)+bf(y)$

Quotient Module:- Let 'M' be A-module and 'N' be sub- of 'M' then $\frac{M}{N} = \{x+N \mid x \in M\}$ is called Quotient Module the binary operations are defined by

i) $(x+N)+(y+N) = (x+y)+N$

ii) $(x+N)\cdot(y+N) = xy+N \quad \forall\, x,y \in M$

Isomorphism:- Let M, N be A-modules a mapping $f: M \to$ called A-module isomorphism if

i) f is onto

ii) f is one-one

iii) f is homomorphism.

Note:- Let $f: M \to N$ be a A-module homomorphism then

1) $\ker f$ is a sub-module of M.

2) $\operatorname{Img} f$ is a sub-module of N.

3) $\text{co-}\ker f \cong \frac{N}{\operatorname{Img} f}$.

Theorem:-1 Consequence of Fundamental Theorem:-

Let $f: M \to N$ be A-module homomorphism and M' be sub-module of $M \subseteq \ker f$ then $\exists$ a homomorphism $\bar{f}: \frac{M}{M'} \to N \ni$ the diagram commutative.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\sigma"'] & & N \\ & \frac{M}{M'} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Proof:- Given that $f: M \to N$ be A-module homomorphism. ie; for any element $x \in M$, we have $f(x) \in N$

Also given that M' be a submodule of M and $M' \subseteq \ker f$.

Now define a mapping $\bar{f}: \frac{M}{M'} \to N$ is given by $f^{-1}(x+M') = f(x)$ where $x \in M$ & $f(x) \in N$

Prove that $\bar{f}$ is well-defined:-

Let us take $x+M', y+M' \in \frac{M}{M'}$ where $x, y \in M$

Suppose $x+M' = y+M'$

$\Rightarrow (x-y) \in M' \subseteq \ker f$

$\Rightarrow x-y \in \ker f$

$\Rightarrow f(x-y) = 0$

$\Rightarrow f(x) - f(y) = 0$

$\Rightarrow f(x) = f(y)$

$\Rightarrow \bar{f}(x+M') = \bar{f}(y+M')$

$\bar{f}$ is onto:-

$\forall f(x) \in N \ni x+M' \in \frac{M}{M'} \ni \bar{f}(x+M') = f(x)$

$\therefore \bar{f}$ is onto.

$\bar{f}$ is a homomorphism:-

consider $\bar{f}(a(x+M') + b(y+M')) = \bar{f}[(ax+M')+(by+M')]$

$= \bar{f}((ax+by)+M')$

$= \bar{f}(ax+by)$

$= a\bar{f}(x) + b\bar{f}(y)$

$\therefore \bar{f}$ is a homomorphism. $= a\bar{f}(x+M') + b\bar{f}(y+M')$

$\therefore \exists$ a homomorphism $\bar{f} : \frac{M}{M'} \to N$

Now we have to show that $\bar{f} \circ \cancel{\sigma} f(x)$

consider $\bar{f} \circ \sigma(x) = \bar{f}(\sigma(x))$

$= \bar{f}(x+M')$

$= f(x)$

$\therefore \bar{f} \circ \sigma = f$

$\therefore$ The diagram is commutative.

Hence proved.

Note:- If we take $M' = \ker f$ then the above theorem is called fundamental theorem of homomorphism.

Proposition:- 2.1 Suppose L, M, N be A module then

i, If $L \supset M \supset N$ then $\frac{L/N}{M/N} \cong \frac{L}{M}$

ii, M_1, M_2 are sub-modules of M then $\frac{M_1+M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$

Proof:- Given that L, M, N be A-modules

i, Given that $L \supset M \supset N$

Now define a mapping $f : \frac{L}{N} \to \frac{L}{M}$ is given by $f(x+N) = x+M \ \forall x \in L$

f is well-defined:-

Let us take $x+N, y+N \in \frac{L}{N} \ni (x+N) = y+N$

$\Rightarrow x-y \in N \subset M$

$\Rightarrow x-y \in M$

$\Rightarrow x+M = y+M$

$\Rightarrow f(x+N) = f(y+N)$

f is onto:-

$\forall x+M \in \frac{L}{M} \ \exists f(x+N) \in \frac{L}{N} \ni f(x+N) = x+M$

$\therefore f$ is onto.

f is homomorphism: $= f((ax+M)+(by+M))$

Consider, $f((a(x+M)+b(y+M)) = f((ax+by)+M)$

$= (ax+by)+N$

$= (ax+N)+(by+N) = a(x+N)+b(y+N)$

$= af(ax+M)+bf(by+M)$

$\therefore f((a(x+M)+b(y+M)) = a\, f(x+M) + b\, f(y+M)$

$\therefore f$ is homomorphism

By fundamental theorem of homomorphism, we have $\frac{L/M}{\ker f} \cong \operatorname{Im} f$

Since f is onto we have $\operatorname{Im} f = \frac{L}{N}$

$\therefore \dfrac{\frac{L}{M}}{\ker f} \cong \frac{L}{N}$

Now we prove that $\ker f = N/M$

$\ker f = \{x+M \in \frac{L}{M} \mid f(x+M) = 0+N\}$

$= \{x+M \in \frac{L}{M} \mid x+N = 0+N\}$

$= \{x+M \in \frac{L}{M} \mid x \in N\}$

$= \{x+M \in \frac{L}{M} \mid x+M \in \frac{N}{M}\}$

$= \frac{N}{M}$

$\therefore \ker f = \frac{N}{M}$

$\therefore \dfrac{\frac{L}{M}}{\frac{N}{M}} \cong \frac{L}{N}$

2) Prove that $\frac{M_1+M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$

ie. Prove that $\frac{M_2}{M_1 \cap M_2} \cong \frac{M_1+M_2}{M_1}$

Now define a mapping $f: M_2 \to \frac{M_1+M_2}{M_1}$ is given by $f(x) = (x+M_1)$

where $x = 0+x \in M_1+M_2$

f is well-defined:

Let us take $x, y \in M_2$, $x = y$

$\Rightarrow x-y = 0 \in M_1$

$\Rightarrow x-y \in M_1$

$\Rightarrow x+M_1 = y+M_1$

$\Rightarrow f(x) = f(y)$

f is onto:- $\forall\ x+M_1 \in \frac{M_1+M_2}{M_1}\ \exists\ x \in M_2 \ni f(x) = x+M_1$

$\therefore f$ is onto.

f is homomorphism:-

consider, $f(ax+by) = (ax+by)+M_1$

$= (ax+M_1)+(by+M_1)$

$= a(x+M_1)+b(y+M_1)$

$= a\,f(x)+b\,f(y)$

$\therefore f(ax+by) = a\,f(x)+b\,f(y)$

$\therefore f$ is homomorphism.

By fundamental theorem of homomorphism, $\frac{M_2}{\ker f} \cong \operatorname{Img} f$

Since f is onto, we have $\frac{M_2}{\ker f} \cong \frac{M_1+M_2}{M_1}$ where $\operatorname{Img} f = \frac{M_1+M_2}{M_1}$

$\Rightarrow \frac{M_2}{\ker f} \cong M_1/f$

Now we have to prove that $\ker f = M_1 \cap M_2$.

$\ker f = \{x \in M_2 \mid f(x) = 0+M_1\}$

$= \{x \in M_2 \mid x+M_1 = 0+M_1\}$

$= \{x \in M_2 \mid x \in M_1\}$

$= \{x \in M_1 \cap M_2\}$

$\therefore \ker f = M_1 \cap M_2$

$\therefore \frac{M_2}{M_1 \cap M_2} \cong \frac{M_1+M_2}{M_1}$ Hence proved.

Annhilator Module:- Let 'M' be A-module and N, P are Sub-modules of M then $(N:P) = \{x \in A / xP \subseteq N\}$ is also sub-module and it is also an ideal of a ring 'A'.

If we take $N = \langle 0 \rangle$ then $(0:P) = \{x \in A \mid xP \subseteq \langle 0 \rangle\}$

ie, $xP = \langle 0 \rangle$ ($\because xP \subseteq \langle 0 \rangle$ & Always $\langle 0 \rangle \subseteq xP \Rightarrow xP = 0$)

ie, $xy = 0\ \forall\ y \in P$

This is called Annhilator ideal of 'P'.

Note:- 1, If $a \in \operatorname{Ann} P$ then $ax = 0\ \forall\ x \in P$

2, An A-module M is called "faithful" $\Leftrightarrow \operatorname{ann} M = 0$

3, If $\operatorname{ann} M = I$ where I is an ideal in a ring 'A' then M is called "faithful $\frac{A}{I}$ module".

Proposition:- 2.2 If M, N, P are A-modules then

i, $Ann(M+N) = (Ann M) \cap (Ann N)$

ii, $(N : P) = Ann\left(\frac{N+P}{N}\right)$

Proof:- Given that

i, M, N be A-modules.

First we have to show that if $M \subseteq N$ then $Ann(N) \subseteq Ann(M)$

Let us take an element $x \in Ann N$ then $xy = 0$ for every $y \in N$

$\Rightarrow xy = 0$ for every $y \in M$

$\Rightarrow x \in Ann M$

$\because M \subseteq N$

$\Rightarrow Ann N \subseteq Ann M$

we know that $M \subseteq M+N$ and $N \subseteq M+N$

$\Rightarrow Ann(M+N) \subseteq Ann M$ and $Ann(M+N) \subseteq Ann N$

$\Rightarrow Ann(M+N) \subseteq (Ann M) \cap (Ann N)$ — (1)

Let us take an element

$a \in (Ann M) \cap (Ann N)$

$\Rightarrow a \in Ann M$ and $a \in Ann N$

$\Rightarrow ax = 0 \ \forall x \in M$ and $ay = 0 \ \forall y \in N$

Now consider $a(x+y) = ax + ay = 0$

$a(x+y) = 0 \ \forall x+y \in M+N$

$\Rightarrow a \in Ann(M+N)$

$\therefore (Ann M) \cap (Ann N) \subseteq Ann(M+N)$ — (2)

From (1) and (2) we have

$Ann(M+N) = (Ann M) \cap (Ann N)$

ii, Let us take an element $a \in (N : P)$

$\Leftrightarrow aP \subseteq N$

$\Leftrightarrow ax \in N \ \forall x \in P$

$\Leftrightarrow ax + N = N = 0 + N$

$\Leftrightarrow a(x+N) = 0 + N$

$\Leftrightarrow a \in Ann\left(\frac{N+P}{N}\right)$

$\therefore (N : P) = Ann\left(\frac{N+P}{N}\right)$ Hence proved.

Note:- 1. Let M, N are A-modules then the direct sum of 'M' and 'N' is

$M \oplus N = \{x+y \mid x \in M, y \in N\}$ is also A-module.

2. The product of M and N is

$M \times N = \{(x, y) \mid x \in M, y \in N\}$ is also A-module.

3. An A-module M is the direct sum of sub modules $M_1, M_2, M_3 \cdots M_n$ can be uniquely expressed as $x_1 + x_2 + x_3 + \cdots + x_n$ where $x_i \in M_i \; \forall i = 1, 2, 3 \cdots n$.

4. An A-module 'M' is said to be finitely generated ie. $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where each M_i is cyclic ie, $M_i = ax_i$ the set $\{x_1, x_2, \cdots x_n\}$ is the generating set for 'n'.

Result:- A-module M, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \Leftrightarrow$

i. $M = M_1 + M_2 + M_3 + \cdots + M_n$

ii. $M_i \cap (M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_i + M_{i+1} + \cdots + M_n) = \langle 0 \rangle$

Proof:- Let 'M' be the A-module and $M_1, M_2, M_3 \cdots M_n$ are sub-modules of M

Let us take $M = M_1 \oplus M_2 \oplus M_3 \oplus \cdots \oplus M_n$

ie, by definition, every element $x \in M$ can be uniquely expressed as $x = x_1 + x_2 + x_3 + \cdots + x_n$ where each $x_i \in M_i \; \forall i = 1, 2, 3 \cdots n$

ie, $M = M_1 + M_2 + M_3 + \cdots + M_n$

ii. Now we have to prove that

$M_i \cap (M_1 + M_2 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n) = \langle 0 \rangle$

Let us take an element

$x \in M_i \cap (M_1 + M_2 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n)$

$\Rightarrow x \in M_i$ and $x \in M_1 + M_2 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n$

$\Rightarrow x = x_i$ and $x = x_1 + x_2 + \cdots x_{i-1} + x_{i+1} + \cdots + x_n$

$\Rightarrow x_i = x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n$

$\Rightarrow x_1 + x_2 + \cdots + x_{i-1} - x_i + x_{i+1} + \cdots + x_n = 0$

$\Rightarrow \sum_{i=1}^{n} x_i = 0$

$\Rightarrow$ each $x_i = 0 \; \forall i = 1, 2, 3 \cdots n$

$\Rightarrow x = 0$

$\therefore M_i \cap (M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n) \subseteq \langle 0 \rangle$

Always we have $\langle 0 \rangle \subseteq M_i \cap (M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n)$

$\therefore M_i \cap (M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n) = \langle 0 \rangle$

Conversely, suppose that

i, $M = M_1 + M_2 + M_3 + \cdots + M_n$ and

ii, $M_i \cap (M_1 + M_2 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n) = \langle 0 \rangle$

prove that $M = M_1 \oplus M_2 \oplus M_3 \oplus \cdots \oplus M_n$

ie, prove that every element $x \in M$ can be uniquely expressed as $x = x_1 + x_2 + x_3 + \cdots + x_n$ where $x_i \in M_i \ \forall \ i = 1, 2, 3 \ldots n$

Suppose $x = x_1 + x_2 + x_3 + \cdots + x_n \ \forall \ x_i \in M_i$
and $x = y_1 + y_2 + y_3 + \cdots + y_n \ \forall \ y_i \in M_i$ $\Big\} \ \forall \ i = 1, 2, 3 \ldots n$

$\Rightarrow x_1 + x_2 + x_3 + \cdots + x_n = y_1 + y_2 + y_3 + \cdots + y_{i-1} + y_i + y_{i+1} + \cdots + y_n$

$\Rightarrow (x_i - y_i) = (y_1 - x_1) + (y_2 - x_2) + (y_{i-1} - x_{i-1}) + (y_{i+1} - x_{i+1}) + \cdots + (y_n - x_n)$ — ①

we have $x_i, y_i \in M_i \ \forall \ i = 1, 2, 3 \ldots n$

where each M_i is a sub-module of M.

$\Rightarrow x_i - y_i \in M_i \ \forall \ i = 1, 2, 3 \ldots n$.

we know that R.H.S. of eqn ① belongs to $M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n$

$\Rightarrow$ L.H.S. of eqn ① is also belongs to $M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n$

ie, $x_i - y_i \in M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n$

but we have $x_i - y_i \in M_i$

$\Rightarrow x_i - y_i \in M_i \cap (M_1 + M_2 + M_3 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n)$

$\Rightarrow x_i - y_i \in \langle 0 \rangle$

$\Rightarrow x_i - y_i = 0 \ \forall \ i = 1, 2, 3 \ldots n$

$\therefore x_1 = y_1, \ x_2 = y_2, \ x_3 = y_3, \ldots x_n = y_n$

$\therefore$ Every element of $x \in M$ can be uniquely expressed as $x_1 + x_2 + x_3 + \cdots + x_n$

$\therefore$ Every element $x \in M$ can be uniquely expressed as $x_1 + x_2 + x_3 + \cdots + x_n$

$\therefore M = M_1 \oplus M_2 \oplus M_3 \oplus \cdots \oplus M_n \Rightarrow$ Hence proved.

Proposition:- 2.3 M is finitely generated A-module $\Leftrightarrow$ M is isomorphic to a quotient of A^n for some +ve integer 'n'.
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Proof:- Let us take 'M' be a finitely generated A-module ie, M is generated by $\{x_1, x_2, x_3 \cdots x_n\}$

$\therefore$ Any element in M can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for $a_1, a_2, a_3 \cdots a_n \in A$

Claim:- M is isomorphic to a quotient of A^n

Define a mapping $f: A^n \to M$ is given by $f(a_1, a_2, a_3 \cdots a_n) = a_1x_1 + a_2x_2 + \cdots + a_nx^n$

i, f is well-defined:- Let us take $(a_1, a_2, a_3 \cdots a_n)$ and $(b_1, b_2, b_3 \cdots b_n) \in A^n$

Suppose $(a_1, a_2, a_3 \cdots a_n) = (b_1, b_2, b_3 \cdots b_n)$

$\Rightarrow a_1 = b_1,\ a_2 = b_2, \cdots a_n = b_n$

$\Rightarrow a_1x_1 = b_1x_1,\ a_2x_2 = b_2x_2, \cdots a_nx_n = b_nx_n$

$\Rightarrow a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1x_1 + b_2x_2 + \cdots + b_nx_n$

$\Rightarrow f(a_1, a_2, a_3 \cdots a_n) = f(b_1, b_2, b_3 \cdots b_n)$

$\therefore f$ is well-defined.

ii, f is onto:- $\forall\ (a_1, a_2, a_3 \cdots a_n) \in A^n\ \exists\ x_1, x_2, x_3 \cdots x_n \in M \ni$
$f(a_1, a_2, a_3 \cdots a_n) = x_1a_1 + a_2x_2 + \cdots + a_nx_n$

f is homomorphism:- consider $f(sa+tb) =$

$$f[s(a_1, a_2, a_3 \cdots a_n) + t(b_1, b_2, b_3 \cdots b_n)]$$
$$= f[(sa_1, sa_2, sa_3 \cdots sa_n) + (tb_1, tb_2, tb_3 \cdots tb_n)]$$
$$= f[sa_1 + tb_1, sa_2 + tb_2, sa_3 + tb_3, \cdots sa_n + tb_n]$$
$$= (sa_1 + tb_1)x_1 + (sa_2 + tb_2)x_2 + \cdots + (sa_n + tb_n)x_n$$
$$= (sa_1x_1 + tb_1x_1) + (sa_2x_2 + tb_2x_2) + \cdots + (sa_nx_n + tb_nx_n)$$
$$= (sa_1x_1 + sa_2x_2 + \cdots + sa_nx_n) + (tb_1x_1 + tb_2x_2 + \cdots + tb_nx_n)$$
$$= s(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + t(b_1x_1 + b_2x_2 + \cdots + b_nx_n)$$

$= s[f(a_1, a_2, a_3 \cdots a_n)] + t[f(b_1, b_2, b_3 \cdots b_n)]$

$= s\, f(a) + t\, f(b)$

$\therefore f(sa+tb) = s\, f(a) + t\, f(b)$

$\therefore f$ is homomorphism.

$\therefore$ By fundamental theorem of homomorphism $\frac{A^n}{\ker f} \cong M$

ie, M is isomorphic to a quotient of A^n

Conversely, suppose that M is isomorphic to quotient of A^n;

ie, $M \cong \frac{A^n}{I}$ where I is an ideal in a ring 'A'.

We have epimorphism $\phi : A^n \to M$

ie, ϕ is onto and homomorphism. $e_1 = (1,0,0 \cdots 0)$ $e_2 = (0,1, \cdots 0)$

Let us denote $e_i = (0, 0, \cdots \underset{i^{th}\ place}{1}, \cdots 0)$ $\forall\, i = 1,2,3 \cdots n$

Let us take $x_i = \phi(e_i)$

Claim:- M is finitely generated ie, M is generated by $\{x_1, x_2, x_3 \cdots x_n\}$

Let us take $x \in M$ $\exists\, y \in A^n \ni \phi(y) = x$ (ϕ onto def)

Now consider, $\phi(y) = \phi(y_1 e_1 + y_2 e_2 + \cdots y_n e_n)$

$= \phi(y_1 e_1) + \phi(y_2 e_2) + \cdots + \phi(y_n e_n)$

$= \phi(y_1)\phi(e_1) + \phi(y_2)\phi(e_2) + \cdots + \phi(y_n)\phi(e_n)$ ($\because y_1, y_2 \cdots$ are scalar)

$= y_1 x_1 + y_2 x_2 + \cdots + y_n x_n$

$\therefore x = y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + x_n y_n$ ($\because \phi(e_i) = x_i$)

$\therefore$ Any element $x \in M$ can be written as linear combination where $y_1, y_2, y_3 \cdots y_n \in A$ $\{x_1, x_2, x_3 \cdots x_n\}$ with the constants $(y_1, y_2, y_3 \cdots y_n)$

$\therefore M$ is generated by $\{x_1, x_2, x_3 \cdots x_n\}$

$\therefore M$ is finitely generated by A-module.

Hence proved.

Free-module:- A cyclic module $M = Ax = \langle x \rangle$ is called free -module. If $Ann(x) = 0$

$\ast \rightarrow$ A cyclic free-module is also called "faithful-Module".

$\ast \rightarrow$ An A-module 'M' is called free-module. If $M = \bigoplus_{\alpha \in I} M_\alpha$ where each M_α is a cyclic free-module.

Proposition:- 2.4 Let 'M' be finitely generated A-module and 'I' be an ideal of a ring 'A', 'ϕ' be A-module endomorphism of M such that $\phi(M) \subseteq IM$ then 'ϕ' satisfies an eqn of the form $\phi^n + a_1\phi^{n-1} + a_2\phi^{n-2} + a_3\phi^{n-3} + \cdots + a_n = 0$ where $a_1, a_2, a_3 \cdots a_n \in I$

Proof:- Given that 'M' be a finitely generated A-module.

ie, M is generated by $\{x_1, x_2, x_3 \cdots x_n\}$ and I be an ideal in a ring 'A'.

Also given that 'ϕ' is endomorphism of 'M' such that $\phi(M) \subseteq IM$

ie, ϕ is a homomorphism of M onto itself.

Now we have $\phi(M) \subseteq IM$

$\Rightarrow \phi(x_i) \in IM \;\forall x_i \in M$

$\Rightarrow \phi(x_i) = \sum_{i,j} a_{ij} x_j \quad (\because a_{ij} \in I,\; i,j = 1,2,3 \cdots n)$

Now applying kronecker-δ on left side we get $\sum_{j=1}^{n} \delta_{ij}\phi(x_i) = \sum_{i,j} a_{ij} x_j$

$\Rightarrow \sum_{j=1}^{n} \delta_{ij}\phi(x_i) - \sum_{i,j} a_{ij}x_j = 0 \quad \left(\because \delta_{ij} \begin{array}{l} = 1 \text{ if } i=j \\ = 0 \text{ if } i \neq j \end{array}\right)$

$\Rightarrow \sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij}) x_j = 0$

This is a homogenous system of 'n' eqn's.

They are $(\delta_{11}\phi - a_{11})x_1 + (\delta_{12}\phi - a_{12})x_2 + \cdots + (\delta_{1n}\phi - a_{1n})x_n = 0$ —(1)

$(\delta_{21}\phi - a_{21})x_1 + (\delta_{22}\phi - a_{22})x_2 + \cdots + (\delta_{2n}\phi - a_{2n})x_n = 0$

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$(\delta_{n1}\phi - a_{n1})x_1 + (\delta_{n2}\phi - a_{n2})x_2 + \cdots + (\delta_{nn}\phi - a_{nn})x_n = 0$

Now substituting the values of δ_{ij}'s we get

$(\phi - a_{11})x_1 + (-a_{12})x_2 + (-a_{13})x_3 + \cdots + (-a_{1n})x_n = 0$

$(-a_{21})x_1 + (\phi - a_{22})x_2 + (-a_{23})x_3 + \cdots + (-a_{2n})x_n = 0$

- - - - - - -

- - - - - - -

$(-a_{n1})x_1 + (-a_{n2})x_2 + \cdots + (\phi - a_{nn})x_n = 0$

The above system of the eqn's can be written in the matrix eq form is $AX = B$

ie, $\begin{bmatrix} \phi - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & \phi - a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \phi - a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Now consider $\text{adj}A \cdot AX = \text{adj}A(0) = 0$ $\quad (\because AX = B)$ Here $B = 0$

$\Rightarrow \text{adj}A(AX) = 0$

$\Rightarrow (\text{adj}A \cdot A)X = 0$

$\Rightarrow |A| X = 0$ $\quad \left(A^{-1} = \frac{1}{|A|}\text{adj}A \quad \frac{1}{A} = \frac{\text{adj}A}{|A|} \right)$

$\Rightarrow |A|(x_i) = 0$

$\Rightarrow |A| = 0 \quad \forall x_i \in M$

On expanding the determinant of 'A' we get $\phi^n + a_1\phi^{n-1} + a_2\phi^{n-2} + \cdots a_n = 0 \quad \forall a_1, a_2, a_3 \cdots a_n \in I$

**** Proposition: 2.5 Let 'M' be finitely generated A-module and 'I' be an ideal of a ring 'A' and $IM = M$ then $\exists$ some element $a \in I$ with the condition $(1+a)M = 0$ (or). 'M' be finitely generated A-module and 'I' be an ideal of a ring 'A' and $IM = M$ then $\exists$ some $a \equiv 1 \pmod{I}$ such that $aM = 0$

Proof:- Given that 'M' be finitely generated A-module and 'I' be an ideal of a ring 'A' and $IM = M$

Let us take an element $x_i \in M$ then $x_i \in IM$

ie, $x_i = \sum_{i,j} a_{ij} \cdot x_j$ where $a_{ij} \in I$

Now applying the Kronecker - Delta on left side we get $\sum_{i,j} \delta_{ij} x_i = \sum_{i,j} a_{ij} \cdot x_i$

$\Rightarrow \sum_{i,j} \delta_{ij} x_i - \sum_{i,j} a_{ij} x_i = 0$

$\Rightarrow \sum_{i,j} (\delta_{ij} - a_{ij}) x_i = 0$

This is a homogeneous system of 'n' equations ie, $(\delta_{11}-a_{11})x_1+(\delta_{12}-a_{12})x_2+\cdots+(\delta_{1n}-a_{1n})x_n=0$

$(\delta_{21}-a_{21})x_1+(\delta_{22}-a_{22})x_2+\cdots+(\delta_{2n}-a_{2n})x_n=0$

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$(\delta_{n1}-a_{n1})x_1+(\delta_{n2}-a_{n2})x_2+\cdots+(\delta_{nn}-a_{nn})x_n=0$

Now substitute the values of δ_{ij}'s we get,

$(1-a_{11})x_1+(-a_{12})x_2+\cdots(-a_{1n})x_n=0$

$(-a_{21})x_1+(1-a_{22})x_2+\cdots+(-a_{2n})x_n=0$

- - - - - - - - - - -

- - - - - - - - - - -

$(-a_{n1})x_1+(-a_{n2})x_2+\cdots+(1-a_{nn})x_n=0$

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now consider $\operatorname{adj}A \cdot AX = \operatorname{adj}A(0)=0$

$\Rightarrow \operatorname{adj}A(AX)=0$

$\Rightarrow (\operatorname{adj}A\cdot A)X=0$

$\Rightarrow |A|X=0$

$\Rightarrow |A|(x_i)=0 \;(\forall\, x_i\in M)$

$\Rightarrow |A|M=0$

On expanding the determinant of 'A' we get $(1+a)M=0$

Hence Proved

***Proposition:-2.6 Nakayama's Lemma:-**

Let 'M' be finitely generated A-module and 'I' be an ideal of a ring 'A' contained in Jacobisan radical of 'A' such that $IM=M$ then $M=0$

Proof:- Given that 'M' be finitely generated A-module and 'I' be an ideal of a ring 'A' $\ni IM=M$

Recall:- proposition - 2.5

By the above recall, we have $(1+a)M=0$

Also given that ideal 'I' is contained in Jacobsan radical of a ring 'A'.

ie, $I \subseteq J(A)$

Now let us take an element $a \in I$

$\Rightarrow a \in J(A)$

Recall:- $a \in J(R) \Leftrightarrow 1+ab$ is a unit in a ring 'R' $\forall\, b \in R$

By the above recall, we have $1+a(1)$ is a unit in a ring 'A',

$\Rightarrow 1+a$ is a unit in a ring 'A'

By definition, $\exists\, b \in A \ni (1+a)b = 1 = b(1+a)$

Now take $M = 1M$

$= b(1+a)\cdot M$

$= b \cdot 0 = 0$

$\therefore M = 0$

Hence proved.

model paper

Corollary-2.7 Let 'M' be finitely generated A-module and 'N' is a Submodule of 'M' and an ideal 'I' is contained in Jacobson radical and $M = N + IM$ then $M = N$

Proof:- Given that 'M' be finitely generated A-module. And 'N' is a submodule of 'M' ie, $N \subseteq M$

To show that $M = N$, it is enough to prove that $M \subseteq N$

Also given that I is contained in Jacobsan radical ie, $I \subseteq$ and $M = N + IM$

Recall:- Nakayama's Lemma

Since M is finitely generated A-module.

$\Rightarrow \frac{M}{N}$ is also finitely generated A-module.

Now consider, $I\left(\frac{M}{N}\right) = I\left(\frac{N+IM}{N}\right)$

$= \frac{I(N+IM)}{N} = \frac{N+IM}{N} = \frac{M}{N}$

$\therefore I\left(\frac{M}{N}\right) = \frac{M}{N}$

By Nakayama's lemma we have $\frac{M}{N} = 0 + N$

Now let us take an element $x + N \in \frac{M}{N}$ where $x \in M$

ie, $x + N = 0 + N$ where $x \in M$

$\Rightarrow x \in N$ where $x \in M$

$\therefore M \subseteq N$

$\therefore M = N$ Hence Proved.

Proposition:- 2.8 Let x_i $(1 \le i \le n)$ be the elements of 'M' whose im-ages in $\frac{M}{mM}$ forms a basis of this vector space then x_i genera-tes M.

Proof:- Given that x_i $(1 \le i \le n)$ be the elements of 'M'.

Let us take 'N' be the submodule of 'M' generated by $\{x_1, x_2, x_3 \cdots x_n\}$

Also given that images of x_i's form a basis of the vector spa-ce $\frac{M}{mM}$

Let us take $\{\bar{x}_1, \bar{x}_2, \bar{x}_3 \cdots \bar{x}_n\}$ be the images of $\{x_1, x_2, x_3 \cdots x_n\}$

where $\bar{x}_i = x_i + mM$ $\forall i = 1, 2, 3 \cdots n$

Now consider, $\Sigma a_i \bar{x}_i = \Sigma a_i (x_i + mM)$ $(a(x+n) = ax + an)$

$$= \Sigma (a_i x_i + mM) \quad (\Sigma m = m)$$

$$= \Sigma a_i x_i + mM$$

$$M = N + mM$$

$\Rightarrow M = N + mM$

Recall:- Corollary-2.7

By the above recall, we have $N = M$

Since N is finitely generated by $\{x_1, x_2, x_3 \cdots x_n\}$, we have 'M' is also finitely generated by $\{x_1, x_2, x_3 \cdots x_n\}$

$\therefore x_i$ generates M.

Hence proved.

* Exact Sequencies *

→ A sequence of A-modules and homomorphisms ....

$M_{i-1} \xrightarrow[f_{i-1}]{} M_i \xrightarrow[f_i]{} M_{i+1} \rightarrow \cdots$ is said to be Exact at M_i if

$\operatorname{img} f_{i-1} = \ker f_i$

The above sequence is called "Long-Exact Sequence" it can be splitted into short exact sequences

i, $0 \rightarrow M' \xrightarrow{f} M$ is exact if 'f' is one-one.

ii, $M \xrightarrow{g} M'' \rightarrow 0$ is exact if 'g' is onto.

iii, $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact if f is one-one, g is onto and $\operatorname{img} f = \ker g$

Theorem:- 2.9 ***** repeated Let $M' \xrightarrow{u} M \xrightarrow{v} M'' \rightarrow 0$ be a sequence of A-modules the above sequence is exact ⟺ ∀ A module 'N' and A-module homomorphism the sequence $0 \rightarrow \operatorname{Homo}(M'',N) \xrightarrow{\bar{v}} \operatorname{Homo}(M,N) \xrightarrow{\bar{u}} \operatorname{Homo}(M',N)$ is exact

Proof:- Let us take the sequence $M' \xrightarrow{u} M \xrightarrow{v} M'' \rightarrow 0$ be a exact sequence.

By definition u is one-one, v is onto and $\operatorname{img} u = \ker v$

prove that $0 \rightarrow \operatorname{Homo}(M'',N) \xrightarrow{\bar{v}} \operatorname{Homo}(M,N) \xrightarrow{\bar{u}} \operatorname{Homo}(M',N)$

ie, prove that $\bar{v}$ is one-one and $\operatorname{img} \bar{v} = \ker \bar{u}$

$$\begin{tikzcd} M' \arrow[r, "u"] \arrow[dr, "fou"'] & M \arrow[r, "v"] \arrow[d, "f"] \arrow[dr, "gov"] & M'' \arrow[d, "g"] \\ & N & N \end{tikzcd}$$

Now define a mapping $\bar{v} : \operatorname{Hom}(M'',N) \rightarrow \operatorname{Hom}(M,N)$ is given by $\bar{v}(g) = gov$

$\bar{u} : \operatorname{Hom}(M,N) \rightarrow \operatorname{Hom}(M',N)$ is given by $\bar{u}(f) = fou$

i, Prove that $\bar{v}$ is one-one:-

Let us take $g \in \operatorname{Hom}(M'',N) \ni \bar{v}(g) = 0$

$$\begin{aligned} &\Rightarrow gov = 0 \\ &\Rightarrow gov(M) = 0 \\ &\Rightarrow g[v(M)] = 0 \\ &\Rightarrow g(M'') = 0 \\ &\Rightarrow g(x) = 0 \ \forall x \in M'' \\ &\Rightarrow g(x) = 0(x) \\ &\Rightarrow g = 0 \end{aligned}$$

∴ $\bar{v}$ is one-one

i, prove that $img\,\bar{v} = ker\,\bar{u}$

ie, prove that $img\,\bar{v} \subseteq ker\,\bar{u}$ and $ker\,\bar{u} \subseteq img\,\bar{v}$

Let us take $f \in img\,\bar{v}$ then $\exists$ an element $g \in hom(M'', N) \ni$

$f = \bar{v}(g)$

$= g \circ v$

Now consider $\bar{u}(f) = f \circ u$

$= (g \circ v) \circ u$

$= g \circ (v \circ u)$

$= g(0)$

$= 0$

$\therefore \bar{u}(f) = 0 \Rightarrow f \in ker\,\bar{u}$

$\therefore img\,\bar{v} \subseteq ker\,\bar{u}$ —(1)

Now we have to show that $ker\,\bar{u} \subseteq img\,\bar{v}$

Let us take an element $f \in ker\,\bar{u}$.

By the definition of $\bar{v}$, we have $\bar{v}(g) = g \circ v$

Also we have $g : M'' \to N$ is given by $g(v(x)) = f(x)\ \forall x \in M$

$g \circ v(x) = f(x)$ (or) $g \circ v = f$

~~Now consider~~, $\therefore$ we have $\bar{v}(g) = g \circ v = f$

$\Rightarrow \bar{v}(g) = f$

ie, $f = \bar{v}(g)$

ie, $f \in img\,\bar{v}$

$\therefore ker\,\bar{u} \subseteq img\,\bar{v}$ —(2)

$\therefore$ From (1) and (2) $ker\,\bar{u} = img\,\bar{v}$

the sequence $0 \longrightarrow Hom(M'', N) \xrightarrow{\bar{v}} Hom(M, N) \xrightarrow{\bar{u}} Hom(M', N)$ is exact

Conversely, suppose that $0 \longrightarrow Hom(M'', N) \xrightarrow{\bar{v}} Hom(M, N) \xrightarrow{\bar{u}} Hom(M', N)$

Prove that $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ is exact

ie, prove that v is onto and $img\,u = ker\,v$

Now we prove that $img\,u = ker\,v$:-

ie, to prove that $img\,u \subseteq ker\,v$ and $ker\,v \subseteq img\,u$

consider $(\bar{u} \circ \bar{v})g = \bar{u}[\bar{v}(g)] = \bar{u}[g \circ v]$

$= (g \circ v) \circ u$

$= g \circ (v \circ u)$

$= g[v \circ u]$

Let us take $[\bar{u} \circ \bar{v}](g) = 0$

$\Rightarrow g(v \circ u) = 0$

$\Rightarrow v \circ u = 0$

$\Rightarrow vou(x) = 0 \ \forall x \in M'$

$\Rightarrow v[u(x)] = 0$

$\Rightarrow u(x) \in \ker v$

$\Rightarrow \operatorname{Img} u \subseteq \ker v$

Now we show that $\ker v \subseteq \operatorname{img} u$

Let us take $x \in \ker v \Rightarrow v(x) = 0$

Already, we have $f(x) = gov(x)$

$= g[v(x)]$

$= g(0)$

$= 0$

$\therefore f(x) = 0$

$\Rightarrow x \in \ker f = \operatorname{img} u$

$\therefore x \in \operatorname{img} u$

$\therefore \ker v \subseteq \operatorname{img} u$

$\therefore \operatorname{Img} u = \ker v$

ii, Now we have to show that v is onto:-

Let us take $N = \dfrac{M''}{\operatorname{img} v}$

we have $\bar{v}$ is one-one iff $\ker \bar{v} = \{0\}$

$\Rightarrow g \in \ker \bar{v} = \{0\}$

$\Rightarrow g \in \{0\}$

$\Rightarrow g = 0$

$\Rightarrow M'' = N$

we know that $\operatorname{img} v = M''$

ie, $v(M) = M''$

ie, v is onto.

$\therefore$ The sequence $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$

Hence proved.

SEC- TENSOR PRODUCT OF MODULES

A - Bilinear:- Let M, N, K be A-modules. A mapping $\theta: M \times N \to K$ is called "A - Bilinear" if

i, $\theta(x_1 + x_2, y) = \theta(x_1, y) + \theta(x_2, y)$

ii, $\theta(ax, y) = a \cdot \theta(x, y)$

iii, $\theta(x, y_1 + y_2) = \theta(x, y_1) + \theta(x, y_2)$

iv, $\theta(x, ay) = a\theta(x, y)$ $\forall a \in A, x, x_1, x_2 \in M$ & $y, y_1, y_2 \in N$

(or)

Let M, N, K be A- modules. A mapping $\theta: M \times N \to K$ is called "A-Bilinear" If for each $y \in N$ the mapping $x \to \theta(x, y)$ of 'M' into 'K' is a A-linear and each $x \in M$ the mapping $y \to \theta(x, y)$ of 'N' into 'K' is "A-linear"

Tensor product:- Let M, N be two A-modules the tensor product of M and N is (T, θ) ; T is A-module and $\theta: M \times N \to T$ is A-bilinear mapping the property that for any A-module "K" and A- bilinear mapping $f: M \times N \to K \ni \bar{f}\theta = f$, $M \times N \to T$. the tensor product of 'M' & 'N' is denoted by $M \otimes N$ (or) $M \otimes_A N$ and $T = \frac{F}{G}$ where 'F' is a free-module defined on $M \times N$ and 'G' is submodule of F.

* * * 16M
Proposition:- 2.10 Let M, N be A-modules then $\exists$ a pair (T, θ) consisting of an A-module T and A- bilinear mapping $\theta: M \times N \to T$ with the following properties given any A-module 'K' and A-bilinear mapping $f: M \times N \to K$ $\exists$ a unique A-linear mapping $\bar{f}: T \to K$ $\ni f = \bar{f} \circ \theta$ moreover if (T, θ) & (T', θ') are two pairs with this property then $\exists$ a unique isomorphism $J: T \to T' \ni J \circ \theta = \theta'$ (or) Tensor product of 'M' & 'N' exists and it is unique upto isomorphism.

repeated

Proof:- Given that M and N be A-modules.

Prove that tensor product of M and N is unique upto isomorphism

Suppose (T, θ) and (T', θ') be the tensor product of A-modules 'M' and 'N'

Consider the diagram $M \times N$

$$\begin{tikzcd} M\times N \arrow[r, "\theta"] \arrow[dr, "\theta'"'] & T \arrow[d, "\phi"] \\ & T' \end{tikzcd} \qquad \begin{tikzcd} M\times N \arrow[r, "\theta'"] \arrow[dr, "\theta"'] & T' \arrow[d, "\psi"] \\ & T \end{tikzcd}$$

By the definition of tensor product $\exists$ a linear mappings 'ϕ' a

$\exists\ \phi\theta = \theta'$ and $\psi\theta' = \theta$

Now consider the diagram
$$\begin{tikzcd} M\times N \arrow[r, "\theta"] \arrow[dr, "\theta"'] & T \arrow[d, "\psi\phi"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} M\times N \arrow[r, "\theta"] \arrow[dr, "\theta"'] & T \arrow[d, "I_T"] \\ & T \end{tikzcd}$$

To show that $\psi\phi = I_T$

Now consider $(\psi\phi)\theta = \psi(\phi\theta)$

$= \psi(\theta')$

$= \theta = I_T(\theta)$

$\psi\phi = I_T$

$\therefore \psi$ and ϕ are inverse to each other. Similarly we get $\phi\psi = I_T$

ie, ψ and ϕ are inverse to each other.

$\therefore \phi$ is an isomorphism of T into T'

$\therefore T \cong T'$

$\therefore$ Tensor product of M and N is unique.

Existence:- Let us take 'F' be a free module defined on $M\times N$ then $F = \{\Sigma a_i(x_i, y_i) \mid a_i \in A, x_i \in M, y_i \in N\}$ is a A-module with respect to formal addition and multiplication.

Let 'G' be the Sub-module of 'F' generated by the element of the type $(x+x', y) - (x-y) - (x', y)$

$(x, ay) - a(x, y)$

$(x, y+y') - (x, y) - (x, y')$

$(ax, y) - a(x, y)$; $x, x' \in M$ and $y, y' \in N$, $a \in A$ becomes zero

Let $T = \frac{F}{G}$ is a A-module

Now define a mapping $\theta : M\times N \to T$ is given by $\theta(x, y) = (x, y) + G$

Now we have to prove that (T, θ) satisfies tensor product condition.

i, Prove that 'θ' is A-bilinear:- ie, prove that

a, $\theta(x_1 + x_2, y) = \theta(x_1, y) + \theta(x_2, y)$

b, $\theta(ax, y) = a\cdot\theta(x, y)$

c, $\theta(x, y_1 + y_2) = \theta(x, y_1) + \theta(x, y_2)$

d, $\theta(x, ay) = a\theta(x, y)$

a, consider $\theta(x_1+x_2, y) = (x_1+x_2, y) + G$

$$= [(x_1, y) + (x_2, y)] + G$$

$$= [(x_1, y) + G] + [(x_2, y) + G]$$

$$= \theta(x_1, y) + \theta(x_2, y)$$

$\therefore \theta(x_1+x_2, y) = \theta(x_1, y) + \theta(x_2, y)$

b, $\theta(ax, y) = (ax, y) + G$

$$= a(x, y) + G$$

$$= a[(x, y) + G]$$

$$= a\,\theta(x, y)$$

$\therefore \theta(ax, y) = a\theta(x, y)$

c, $\theta(x, y_1+y_2) = [(x, y_1) + (x, y_2)] + G$

$$= (x, y_1) + G + [(x, y_2) + G]$$

$$= \theta(x, y_1) + \theta(x, y_2)$$

$\therefore \theta(x, y_1+y_2) = \theta(x, y_1) + \theta(x, y_2)$

d, $\theta(x, ay) = (x, ay) + G$

$$= a(x, y) + G$$

$$= a\,\theta(x, y)$$

$\therefore \theta(x, ay) = a\,\theta(x, y)$

$\therefore \theta$ is "A- bilinear".

Consider any A-module 'k' and a A-bilinear $f: M\times N \longrightarrow K$ then the mapping 'f' can be extended to A-linear mapping $f_1: F \longrightarrow K$

$f_1(\Sigma a_i(x_i, y_i)) = \Sigma a_i f(x_i, y_i)$

Also f_1 vanishes on G. ($\because$ By def., of 'G' and a bilinear of 'f')

Finally, we claim that this mapping f_1 induces to A-linear mapping ~~its given by A-li~~ ~~-near mapping~~ $\bar{f}: T \rightarrow K$ is given by

$$\bar{f}(x, y) + G = f(x, y)$$

i, prove that $\bar{f}[(x_1, y_1) + G + (x_2, y_2) + G] = \bar{f}[(x_1, y_1) + G] + \bar{f}[(x_2, y_2) + G]$

Consider $\bar{f}[(x_1, y_1) + G + (x_2, y_2) + G] = \bar{f}((x_1, y_1) + (x_2, y_2) + G)$

$$= f[(x_1, y_1) + (x_2, y_2)]$$

$$= f(x_1, y_1) + f(x_2, y_2)$$

$$= (\bar{f}(x_1, y_1) + G) + (\bar{f}(x_2, y_2) + G)$$

ii, Prove that $\bar{f}[a(x_1, y_1) + G] = a\bar{f}[(x_1, y_1) + G]$

consider, $\bar{f}[a(x_1, y_1) + G] = f[a(x_1, y_1)]$

$$= a f(x_1, y_1)$$

$$= a\bar{f}[(x_1, y_1) + G]$$

$\therefore \bar{f}$ is a A-linear mapping.

Now we have to show that $\bar{f}\theta = f$:-

we have $\theta : M \times N \to T$ and $\bar{f} : T \to K$

$\bar{f}\theta : M \times N \to K$

Let us take $(x,y) \in M \times N$

Now consider $\bar{f}\theta(x,y) = \bar{f}[\theta(x,y)]$

$= \bar{f}[(x,y) + G]$

$= f(x,y)$

$\therefore \bar{f}\theta = f$

$\therefore$ Tensor product of 'M' and 'N' exist and it is unique upto -morphism.

Hence proved.

Theorem:-13 Let M, N and P be A-modules then

i, $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$

ii, $M \otimes N \cong N \otimes M$

iii, $M \otimes A \cong M \cong A \otimes M$

Proof:- Given that M, N and P be A-modules

i, Define a mapping $\theta : (M \otimes N) \times P \longrightarrow (M \otimes N) \otimes P$ is given by $\theta[x \otimes y, z] = x \otimes y \otimes z$ is well defined and it is a A-bilin mapping.

This mapping can be extended to the A-linear mappin $f : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P)$ is given by $f[(x \otimes y) \otimes z] =$ ... $(y \otimes z)$

This is an homomorphism and its inverse mapping is given by $g : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$ is given by

$g(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$.

$\therefore g$ and f are inverses to each other.

$\therefore (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$

ii, Define a mapping $\theta : M \times N \longrightarrow M \otimes N$ is given by $\theta(x \times y) =$ ... is well-defined and it is a A-bilinear mapping.

The mapping can be extended to the A-linear mapp $f : M \otimes N \longrightarrow N \otimes M$ is given by $f(x \otimes y) = y \otimes x$

This is an homomorphism and its inverse mapping also given by $g(y \otimes x) = x \otimes y$

$\therefore$ g and f are inverses to each other.

$\therefore M \otimes N \cong N \otimes M$

ii) Define a mapping $\phi : A \otimes M \to M$ is given by $\phi(a \otimes x) = ax$ where $a \otimes x \in A \otimes M$

Now we have to show that the mapping 'ϕ' preserves addition and scalar multiplication.

let us take $a \otimes x, a \otimes y \in A \otimes M$

consider, $\phi[(a \otimes x) + (a \otimes y)] = \phi[a \otimes (x+y)]$

$= a(x+y)$

$= ax + ay$

$= \phi(a \otimes x) + \phi(a \otimes y)$

consider, $\phi[r(a \otimes x)] = \phi[ra \otimes x]$

$= ra \cdot x$

$= r(ax)$

$= r(\phi(a \otimes x))$

$\therefore \phi$ preserves addition and scalar multiplication.

Now define a mapping $\psi : M \to A \otimes M$ is defined by $\psi(x) = 1 \otimes x$ where $x \in M$ and $1 \in A$

Clearly 'ψ' is well-defined and it is a A-linear mapping and also preserves addition and scalar multiplication.

Now we have to show that $\phi\psi = I_M$ and $\psi\phi = I_{A \otimes M}$

Now consider, $\phi\psi(x) = \phi[\psi(x)]$

$= \phi[1 \otimes x]$

$= 1 \cdot x = x = I_M(x)$

$\therefore \phi\psi = I_M$

lly we can prove $\psi\phi = I_{A \otimes M}$

$\therefore \phi$ and ψ are inverses to each other.

$\therefore \phi$ is an isomorphism.

ie, $A \otimes M \cong M$

lly we can prove that $M \otimes A \cong M$

$\therefore A \otimes M \cong M \cong M \otimes A$

Hence proved

Theorem:- 14 If M is finitely generated S-module and 'S' is finitely generated A-module then M is finitely generated A-module.

proof:- Let us take M is finitely generated S-module.

ie, take $\{x_1, x_2, x_3 \dots x_n\}$ be the generators of 'M' over 'S'.

Also, given that 'S' be finitely generated A-module.

Let us take $\{y_1, y_2, y_3 \dots y_n\}$ be the generators of 's' over 'A'.

Taken an element $x \in M$ then $x = \sum a_i x_i$ where $a_i \in S$ —(1)

Now $a_i \in S$ and $\{y_1, y_2, y_3 \dots y_n\}$ be generating set for 'S' over A

ie, $a_i = \sum a_{ij} \cdot y_j$ —(2) where $a_{ij} \in A$

Substituting (2) in (1) we get $x = \sum\sum a_{ij} y_j x_i$

$= \sum\sum a_{ij} x_i y_j$ where $a_{ij} \in A$

$\therefore \{x_i \cdot y_j \mid i = 1, 2, 3 \dots n, j = 1, 2, 3 \dots n\}$ be the generating set for M over A.

$\therefore$ M is finitely generated A-module

Hence proved.

Definition:- For any A-module 'M', $S \otimes_A M = M_S$ is S-module and it is defined by $a(s_i \otimes x) = as_i \otimes x$ where $s_i \in S$, $x \in M$ and $a \in$ here 'S' is multiplicatively closed subset of a ring 'A'.

Theorem:-15 If 'M' is finitely generated A-module then $S \otimes_A M$ is finitely generated S-module & MS is finitely generated S-module

proof:- Given that M is finitely generated A-module

Let us take $\{x_1, x_2, x_3 \dots x_n\}$ be the generators of M over

ie, for any $x \in M$, we have $x = \sum a_i x_i$ where $a_i \in A$

prove that $S \otimes_A M$ is finitely generated S-module.

Let us take $s \otimes x \in S \otimes M$ where $s \in S$ and $x \in M$

Now consider, $s \otimes x = s \otimes \sum a_i x_i$

$= \sum s a_i \otimes x_i$

$= \sum s a_i (1 \otimes x_i)$ where $s a_i \in S$

$\therefore \{1 \otimes x_i \mid i = 1, 2, 3 \dots n\}$ is the generating set for $S \otimes M$ over 'S'.

$\therefore S \underset{A}{\otimes} M$ is finitely generated S-module.

Hence proved

→ Flat:- An A-module 'M' is called "Flat" if every exact Sequence of A-modules $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$, the induced sequence $0 \longrightarrow M \otimes N' \xrightarrow{f^*} M \otimes N \xrightarrow{g^*} M \otimes N'' \longrightarrow 0$ is also exact.

Theorem:-16 For any A-module 'M', the following are equivalent

i, M is a flat A-module.

ii, for any exact sequence of A-module $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$, $0 \longrightarrow M \times N' \xrightarrow{f^*} M \times N \xrightarrow{g^*} M \times N'' \longrightarrow 0$ is exact

the sequence $0 \longrightarrow M \times N' \xrightarrow{f^*} M \times N$ is injective.

iii, if $f: N' \longrightarrow N$ is injective then $f^*: M \times N' \longrightarrow M \times N$ is injective.

iv, if $f: N' \longrightarrow N$ is injective with N, N' are finitely generated then $f^*: M \times N' \longrightarrow M \times N$ is injective.

Proof:- Given that 'M' be a A-module

i ⇒ ii :- Let us take 'M' is a flat A-module.

By definition, for any exact sequence of A-modules $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$, the induced sequence $0 \longrightarrow M \times N' \xrightarrow{f^*} M \times N \xrightarrow{g^*} M \times N'' \longrightarrow 0$ is also exact.

ii ⇒ iii :- Let us take $f: N' \longrightarrow N$ is injective.

⇒ The ~~Sequence~~ $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$ is a exact Sequence

By ii, we have $0 \longrightarrow M \times N' \xrightarrow{f^*} M \times N \xrightarrow{g^*} M \times N'' \longrightarrow 0$ is also exact

$\therefore f^*: M \times N' \longrightarrow M \times N$ is injective.

iii ⇒ ii :- If $f: N' \longrightarrow N$ is injective then $f^*: M \times N' \longrightarrow M \times N$ is also injective.

Let us take $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$ be an exact Sequence

⇒ f is one-one, g is onto and $\operatorname{img} f = \ker g$

⇒ f^* is one-one, g^* is onto and $\operatorname{img} f^* = \ker g^*$

$\therefore 0 \longrightarrow M \times N' \xrightarrow{f^*} M \times N \xrightarrow{g^*} M \times N'' \longrightarrow 0$ is also exact.

iii ⇒ iv :- Let us take $f: N' \longrightarrow N$ is injective also take N, N' are finitely generated A-modules.

$\therefore f^*: M \times N' \longrightarrow M \times N$ is also injective.

iv $\Rightarrow$ iii:- Let us take $f: N' \longrightarrow N$ is injective, where N, N' a
finitely generated.

Claim:- $f^*: M \times N' \longrightarrow M \times N$ is injective.

Let us take an element $\sum x_i \otimes y_i \in \ker f^* \Rightarrow f^*(\sum x_i \otimes y_i) = 0$

$\Rightarrow \sum f^*(x_i \otimes y_i) = 0$

Recall:- For any exact sequence of A-module $0 \rightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \rightarrow$
the sequence $0 \longrightarrow M \times N' \xrightarrow{f^*} M \times N \xrightarrow{g^*} M \times N'' \longrightarrow 0$ is exact.

In that sequence, f^* and g^* are defined, by $f^*(x \otimes y) = x \otimes f^*(y) = x \otimes f(y)$

and $g^*(x \otimes z) = x \otimes g^*(z) = x \otimes g(z)$

Let us take N_1' be the sub-module of N' and it is generated by $\{y_i\}$

Let us take $t = \sum x_i \otimes y_i \in M \otimes N'$

Now take $f^*(x_i \otimes y_i) = 0$

$\Rightarrow x_i \otimes f(y_i) = 0$ (or) $\sum x_i \otimes f(y_i) = 0$

$\therefore \exists$ finitely generated sub-module N_1 of N containing $f(N_1') \ni$
$\sum x_i \otimes f(y_i) \in M \otimes N_1$ is zero

ie, $f(t) = 0$

$f: N' \longrightarrow N$

$N_1' \quad N_1$ sub module

$f(N_1') \subseteq N_1 \subseteq N$

$\Rightarrow t = 0$ ($\because f$ is one-one)

$\Rightarrow \sum x_i \otimes y_i = 0$

$\Rightarrow \ker f^* = 0$

$\therefore f^*$ is one-one.

ie, f^* is injective.

$\Rightarrow f^*: M \times N' \rightarrow M \times N$ is Injective.

Hence proved.

Corollary:- 17 Let $f: A \rightarrow S$ be a ring homomorphism if M is a flat A-module then $S \underset{A}{\otimes} M$ is a S-flat module.

(or)

If M is a flat A-module then M_S is a flat S-module.

Proof:- Given that $f: A \rightarrow S$ be a ring homomorphism. Let us take M is a flat A-module.

By definition, Let $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \rightarrow 0$ be an exact sequence of A-modules can be induced to $0 \rightarrow M \times N' \xrightarrow{f^*} M \times N \xrightarrow{g^*} M \times N'' \rightarrow 0$

is an exact sequence.

Recall:- $S \otimes M \cong M$ (Theorem - 13 (iii))

$\Rightarrow 0 \longrightarrow (S \otimes M) \times N' \longrightarrow (S \otimes M) \times N \longrightarrow (S \otimes M) \times N'' \longrightarrow 0$ is an exact sequence.

$\therefore$ By definition, $S \otimes M$ is a flat S-module

ie, M_S is a flat S-module.

Hence proved.

Definition:- A non-empty set 'p' is called an algebra over the ring 'A' (or) 'B' is an A-algebra if it has binary operations '+' and '.' defined on it and it satisfies the following conditions.

i, B is a ring

ii, B is a A-module.

iii, $a x(y) = a(xy)$ (or) $a(xy) = ax(y)$ $\forall a \in A, x, y \in B$